FAR BEYOND

MAT122

Area Between the Curves

Stony Brook University

Integrating with Absolute Value

ex: Evaluate $\int_{-4}^{5} |x|\, dx$ Start by writing as a piecewise function: $f(x) = \begin{cases} -x, & x < 0, \\ x, & x \geq 0. \end{cases}$

$$= \int_{-4}^{0} -x\, dx + \int_{0}^{5} x\, dx$$

$$= -\frac{x^2}{2}\Big|_{-4}^{0} + \frac{x^2}{2}\Big|_{0}^{5}$$

$$= -\frac{1}{2}\left(0^2 - (-4)^2\right) + \frac{1}{2}\left(5^2 - 0^2\right)$$

$$= -\frac{1}{2}(-16) + \frac{1}{2}(25)$$

$$= \frac{16}{2} + \frac{25}{2}$$

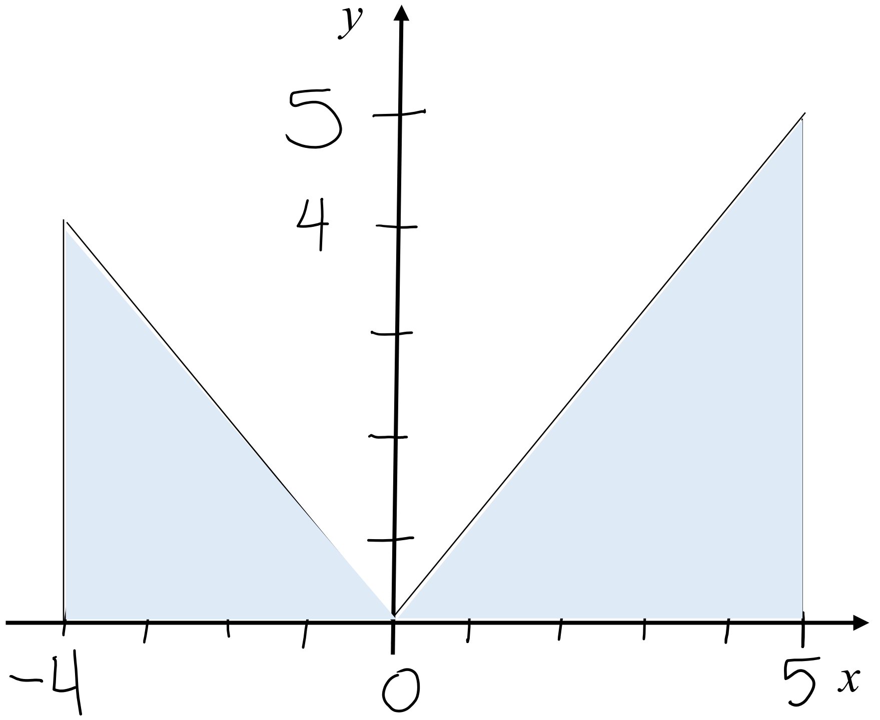
$$= \boxed{\frac{41}{2}}$$

Method #2: use standard shapes to find area

$$= \int_{-4}^{0} |x|\, dx + \int_{0}^{5} |x|\, dx$$

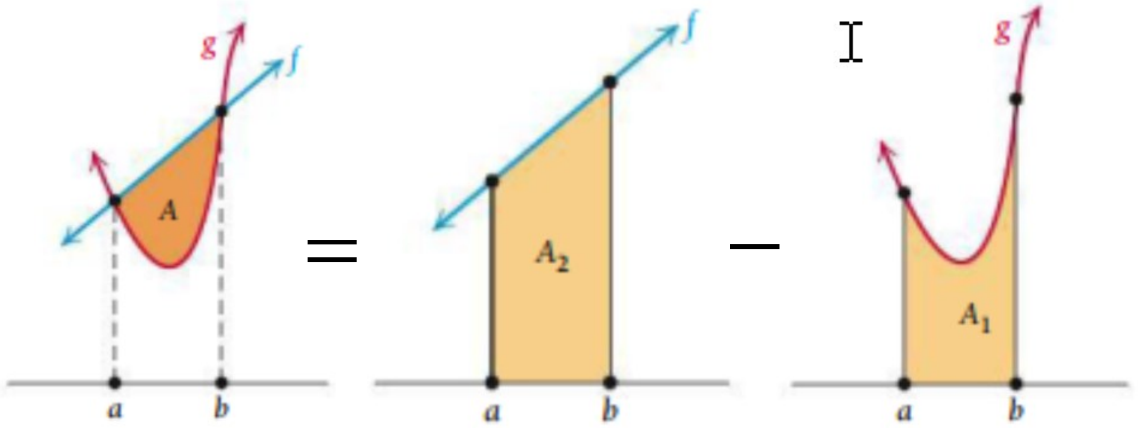
$$= \frac{1}{2}\cdot 4\cdot 4 + \frac{1}{2}\cdot 5\cdot 5 = \frac{16}{2} + \frac{25}{2} = \boxed{\frac{41}{2}}$$

$$A_{\Delta} = \frac{1}{2}bh$$



Area of a Region Bound by Multiple Graphs

$$\int_a^b \left(f(x) - g(x)\right)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$



ex. Find the area of the region bound by the graphs $f(x) = 2x+1$ and $g(x) = x^2+1$.

First, find the bounds of integration:

(set functions equal to each other)

$$f(x) = g(x)$$

$$2x+1 = x^2+1$$

solve algebraically

$$0 = x^2 - 2x$$

$$0 = x(x-2)$$

$x = 0$ lower bound $x = 2$ upper bound

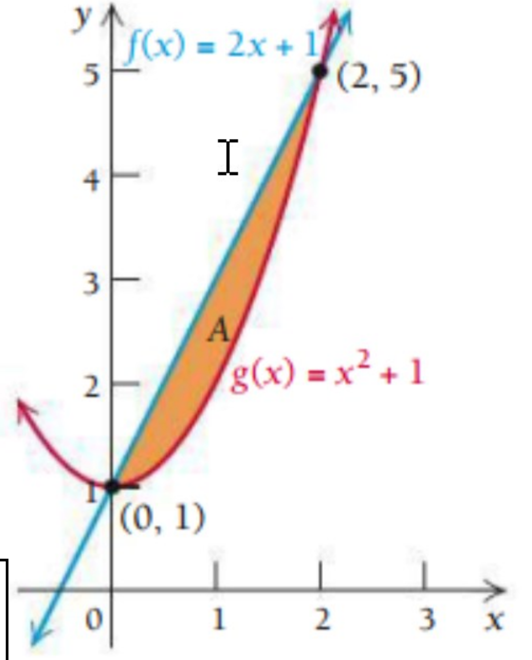
since $f(x) \geq g(x)$ on $[0,2]$ put f first in integral

$$\int_0^2 \left[(2x+1) - (x^2+1)\right] dx$$

$$= \int_0^2 \left[2x + 1 - x^2 - 1\right] dx$$

$$= \int_0^2 (2x - x^2)\, dx$$

$$= 2\frac{x^2}{2} - \frac{x^3}{3} \Big|_0^2 = 2^2 - \frac{2^3}{3} - 0 = 4 - \frac{8}{3} = \boxed{\frac{4}{3}}$$



Area Between Two Curves – Example #2

ex. Use an integral to find the area of the region bounded above by $y = e^x$, bounded below by $y = x$ and bounded on the sides by $x = 0$ and $x = 1$.

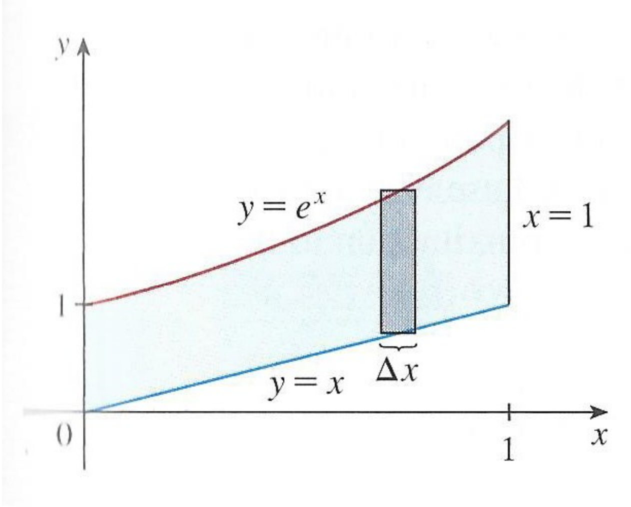
$$\int_0^1 \left(\overset{Y_T}{e^x} - \overset{Y_B}{x}\right)dx$$

$$= \left(e^x - \frac{x^2}{2}\right)\Bigg|_0^1$$

$$= e^1 - \frac{1}{2} - \left(e^0 - 0\right) \quad (e^0 = 1)$$

$$= e - \frac{1}{2} - 1$$

$$= \boxed{e - \frac{3}{2}}$$



$$\int_a^b \left(Y_{Top} - Y_{Bottom}\right)dx$$

Area Between Two Curves – Do

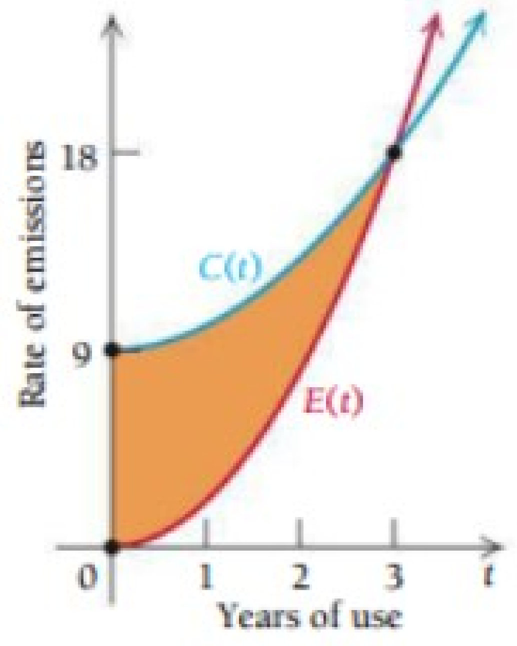
$$\int_a^b (Y_T - Y_B)\,dx$$

ex. Find the area of the region bounded by $y = x,\ \ y = x^3, x = 0,\ \ x = 1.$

$$= \boxed{\frac{1}{4}}$$

Area Between Two Curves – Application

ex. A student develops an engine believed to meet state emission standards.
The new engine's emission rate is given by $E(t) = 2t^2$
where E is in billions of pollution particles per year and t is time in years.
The emission rate of a conventional engine is given by $C(t) = 9 + t^2$.



$$\int_a^b (Y_T - Y_B)\,dx$$

At what time will the emission rates be the same?

$$E(t) = C(t)$$
$$2t^2 = 9 + t^2$$
$$t^2 = 9$$
$$t^2 - 9 = 0$$
$$(t+3)(t-3) = 0 \Rightarrow t = 3 \text{ or } t = -3$$

invalid solution

What reduction in emissions results from the use of the student's engine over first 3 years?

$$\int_0^3 [(9+t^2) - 2t^2]\,dt$$
$$= \int_0^3 (9 - t^2)\,dt$$
$$= \left(9t - \frac{t^3}{3}\right)\Bigg|_0^3 = \left(9 \cdot 3 - \frac{3^3}{3}\right) = 27 - 9 = \boxed{18}$$

Interpretation:
Over 3 years, student's engine reduces emissions by 18 billion pollution particles.